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Superradiance and black hole bomb in five-dimensional minimal ungauged supergravity

Alikram N. Aliev

Faculty of Engineering and Architecture, Yeni Yüzyıl University,
Cevizlibağ-Topkapı, Istanbul, 34010 Turkey

E-mail: alikram.n.aliev@gmail.com

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Abstract. We examine the black hole bomb model which consists of a rotating black hole of five-dimenensional minimal ungauged supergravity and a reflecting mirror around it. For low-frequency scalar perturbations, we find solutions to the Klein-Gordon equation in the near-horizon and far regions of the black hole spacetime. To avoid solutions with logarithmic terms, we assume that the orbital quantum number l takes on nearly, but not exactly, integer values and perform the matching of these solutions in an intermediate region. This allows us to calculate analytically the frequency spectrum of quasinormal modes, taking the limits as l approaches even or odd integers separately. We find that all l modes of scalar perturbations undergo negative damping in the regime of superradiance, resulting in exponential growth of their amplitudes. Thus, the model under consideration would exhibit the superradiant instability, eventually behaving as a black hole bomb in five dimensions.

Keywords: gravity, absorption and radiation processes

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Contents

1 Introduction

The phenomenon of *superradiance* through which waves of certain frequencies are amplified when interacting with a medium has long been known in both classical and quantum non-gravitational systems. The quantum aspect of this phenomenon traces back to the so-called Klein paradox [1, 2] whose subsequent resolution revealed the existence of superradiant boson (not fermion) states in the background of strong electromagnetic fields (see e.g. [3] and references therein). The superradiant effect also arises in many classical systems moving through a medium with the linear velocity that exceeds the phase velocity of waves under consideration. As early as 1934 it was known that the reflection of sound waves from the boundary of a medium, which moves with supersonic velocity, occurs with amplification [4]. Subsequently, examples of such an amplification were found in a number of cases; for instance, in the motion of carriers in an elastic piezoelectric substance [5] as well as in the motion of a conducting liquid in a resonator [6].

Zel'dovich first realized that the superradiant condition can be fulfilled in a rotational case as well [7, 8]. Suggesting that for a wave of frequency ω and angular momentum m, the angular velocity Ω of a body can exceed the angular phase velocity ω/m of the wave, $\Omega > \omega/m$, he demonstrated the amplification of waves reflected from a rotating and conducting cylinder. In addition, Zel'dovich put forward the idea that a semitransparent mirror surrounding the cylinder could provide exponential amplification of waves. He also anticipated that the phenomenon of superradiance and the process of exponential amplification of waves would occur in the field of a Kerr black hole. The black hole superradiance was independently predicted by Misner [9], who pointed out that certain modes of scalar waves scattered off the Kerr black hole undergo amplification. Possible applications of the superradiant mechanism were explored by Press and Teukolsky [10]. In particular, by locating a spherical mirror around a rotating black hole they pointed out that such a system would eventually develop a strong instability against exponentially growing modes in the superradiant regime, thus creating a *black hole bomb*.

The quantitative theory of superradiance for scalar, electromagnetic and gravitational waves in the Kerr metric was developed in classic papers by Starobinsky [11] and Starobinsky and Churilov [12] (see also [13, 14]). The existence of superradiance is intimately related to the salient feature of the Kerr metric; the timelike Killing vector that defines the energy with respect to asymptotic observers becomes spacelike in the region located outside the horizon, called the *ergoregion*. This in turn entails the possibility of negative energy states within the ergoregion, underpinning the physical interpretation of the superradiant effect. Scattering a wave off a rotating black hole may cause fluctuations of the negative energy states, resulting in the negative energy flux into the black hole [15]. As a consequence, the scattered wave becomes amplified, by conservation of energy. It should be noted that there is no superradiance for fermion modes in the Kerr metric, as shown by detailed calculations in [16, 17].

The black hole superradiance on its own has only a conceptual significance, showing the possibility of the extraction of rotational energy from the black hole due to the wave mechanism. However, it has played a profound role in addressing the stability issues of rotating black holes in general relativity, against small external perturbations. Developments in this direction have revealed that rotating black holes are stable to massless scalar, electromagnetic and gravitational perturbations [13, 14]. On the contrary, it appeared that small perturbations of a massive scalar field grow exponentially in the superradiant regime, creating the instability of the system, the black hole bomb effect [18–20]. The physical reason underlying this effect is that the motion of a massive particle around a rotating black hole may occur in stable circular orbits [21] (see also a recent paper [22]). Thus, to view the instability one can imagine a wave-packet of the massive scalar field moving in these orbits and forming "bound states" in the well of the effective potential of the motion. Though the potential barrier keeps the wave-packet bound states in the well from escaping to infinity, but from quantum-mechanical point of view they would tunnel through the barrier into the horizon. As a consequence, the bound states in the well become *quasinormal* with complex frequencies whose imaginary parts in the superradiant regime determine the growth rate of the wave-packet modes. It is clear that the runaway behavior of such modes between the potential well and the horizon would result in their continuous reamplification, thereby causing the instability.

Another realization of the black hole bomb effect occurs in anti-de Sitter (AdS) spacetimes. This is due to the fact that in the regime of superradiance, the timelike boundary of the AdS spacetime plays the role of a resonant cavity between the black hole and spatial infinity. In [23], it was argued that small rotating AdS black holes in five dimensions may exhibit the superradiant instability against external perturbations. The authors of works [24, 25] were the first to develop these arguments further by using both analytical and numerical approaches. Elaborating on the black hole bomb effect of Press and Teukolsky in four dimensions, they pointed out that its realization crucially depends on the distance at which the mirror must be located. Thus, for the superradiant modes to be excited there exists a critical radius and below this radius the system is stable. These results allow one to clarify the instability of small Kerr-AdS black holes, as discussed in [25]. Continuing this line of investigation in five dimensions, the superradiant instability of small rotating charged AdS black holes was considered in [26]. Meanwhile, the case of arbitrarily higher dimensions for small Reissner-Nordström-AdS black holes has recently been studied in [27]. In particular, it was noted that for some values of the orbital quantum number l, which can occur in odd spacetime dimensions, the analytical approach of [26] fails being responsible for the seeming absence of the superradiant instability for certain modes. Detailed numerical calculations have shown that the superradiant instability exists in all higher dimensions and with respect to all modes of scalar perturbations [27].

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In this paper, we wish to embark on a further exploration of the superradiant instability for rotating black holes in five dimensions. We consider the black hole bomb model for scalar perturbations, which consists of a rotating black hole of five-dimenensional minimal ungauged supergravity and a reflecting mirror around it. In section 2 we begin by discussing the defining properties of the spacetime metric for the black hole under consideration. Here we present remarkably simple formulas for the coordinate angular velocities of locally non-rotating observers. These formulas reveal the "bi-dragging" property of the black hole at large distances and reduce to its angular velocities as one approaches the horizon. Next, we introduce a corotating Killing vector field which is tangent to the null geodesics of the horizon and calculate the surface gravity and the electrostatic potential of the horizon. In section 3 we discuss the separated radial and angular parts of the Klein-Gordon equation for a charged massless scalar field and derive the threshold inequality for superradiance. Focusing on low-frequency perturbations, in section 4 we find solutions to the radial wave equation by dividing the spacetime into the near-horizon and far regions. To avoid solutions with logarithmic terms, we then assume that the orbital quantum number l is an approximate integer and perform the matching of these solutions in an intermediate region. In section 5 we calculate the frequency spectrum of quasinormal modes in the black hole-mirror system, taking the limits as l approaches even or odd integers separately. Here we show that in the regime of superradiance, the black hole-mirror system exhibits instability to all l modes of scalar perturbations. In section 6 we end up with a discussion of our results.

2 The metric

The general solution to five-dimensional minimal gauged supergravity that describes charged and rotating black holes with two independent rotational symmetries was found by Chong, Cvetič, Lü and Pope (CCLP) [28]. In the case of ungauged supergravity (the vanishing cosmological constant) it is given by the metric

$$\begin{aligned} ds^2 = &-\left(dt - a\sin^2\theta\, d\phi - b\cos^2\theta\, d\psi\right)\Big[f\left(dt - a\sin^2\theta\, d\phi\ - b\cos^2\theta\, d\psi\right) \\ &+\frac{2Q}{\Sigma}\left(b\sin^2\theta\, d\phi + a\cos^2\theta\, d\psi\right)\Big] + \Sigma\left(\frac{r^2dr^2}{\Delta} + d\theta^2\right) \\ &+\frac{\sin^2\theta}{\Sigma}\left[a\, dt - (r^2+a^2)\, d\phi\right]^2 + \frac{\cos^2\theta}{\Sigma}\left[b\, dt - (r^2+b^2)\, d\psi\right]^2 \\ &+\frac{1}{r^2\Sigma}\left[ab\, dt - b(r^2+a^2)\sin^2\theta\, d\phi - a(r^2+b^2)\cos^2\theta\, d\psi\right]^2, \end{aligned} \tag{2.1}$$

where the metric functions are given by

$$\begin{aligned} f &= \frac{\left(r^2+a^2\right)\left(r^2+b^2\right)}{r^2\Sigma} - \frac{2M\Sigma - Q^2}{\Sigma^2}, \qquad \Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \\ \Delta &= \left(r^2+a^2\right)\left(r^2+b^2\right) + 2ab\, Q + Q^2 - 2Mr^2, \end{aligned} \tag{2.2}$$

the parameters M and Q are related to the physical mass and electric charge of the black hole, whereas a and b are two independent rotation parameters. The metric determinant is given by

$$\sqrt{-g} = r\Sigma\sin\theta\,\cos\theta\,. \tag{2.3}$$

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It is straightforward to check that this metric and the two-form field $F = dA$, where

$$A = -\frac{\sqrt{3}\,Q}{2\Sigma}\,\left(dt - a\sin^2\theta\,d\phi - b\cos^2\theta\,d\psi\right) \tag{2.4}$$

is the potential one-form of the electromagnetic field supporting the metric, satisfy the equation of motions derived from the action of five-dimensional minimal ungauged supergravity

$$S = \int d^5x\sqrt{-g}\left(R - \frac{1}{4}\,F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{12\sqrt{3}}\,\epsilon^{\mu\nu\alpha\beta\lambda}F_{\mu\nu}F_{\alpha\beta}A_{\lambda}\right). \tag{2.5}$$

The locations of the outer and inner horizons of the black hole are determined by the real roots of the equation $\Delta = 0$. Thus, we find that

$$r_{\pm}^2 = \frac{1}{2}\left[-(a^2+b^2-2M) \pm \sqrt{(a^2+b^2-2M)^2 - 4(ab+Q)^2}\right], \tag{2.6}$$

where r_+^2 corresponds to the radius of the outer (the event) horizon, while r_-^2 gives the radius of the inner Cauchy horizon. It follows that for the extremal horizon, $r_+^2 = r_-^2$, there exist two simple relations between the parameters of the black hole, which are given by

$$M = \frac{(a+b)^2}{2} + Q\,, \qquad or \qquad M = \frac{(a-b)^2}{2} - Q\,. \tag{2.7}$$

In the following we will also need the inverse components of metric (2.1), which are given by

$$\begin{aligned}
g^{00} &= -1 + \frac{(r^2+b^2)\left[Q^2 - 2M(r^2+a^2)\right] + a^2Q^2}{\Delta\Sigma}\,, \qquad g^{11} = \frac{\Delta}{r^2\Sigma}\,,\\
g^{22} &= \frac{1}{\Sigma}\,, \qquad g^{33} = \frac{1}{\Sigma}\left[\frac{1}{\sin^2\theta} + \frac{(r^2+b^2)(b^2-a^2) - 2b(aQ+bM)}{\Delta}\right],\\
g^{44} &= \frac{1}{\Sigma}\left[\frac{1}{\cos^2\theta} + \frac{(r^2+a^2)(a^2-b^2) - 2a(bQ+aM)}{\Delta}\right], \qquad g^{34} = -\frac{2abM + (a^2+b^2)\,Q}{\Delta\Sigma}\,,\\
g^{03} &= -\frac{(2aM+bQ)(r^2+b^2) - aQ^2}{\Delta\Sigma}\,, \qquad g^{04} = -\frac{(2bM+aQ)(r^2+a^2) - bQ^2}{\Delta\Sigma}\,.
\end{aligned} \tag{2.8}$$

It is easy to see that the stationary and bi-azimuthal isometries of this metric are described by three commuting Killing vectors

$$\xi_{(t)} = \frac{\partial}{\partial t}\,, \qquad \xi_{(\phi)} = \frac{\partial}{\partial\phi}\,, \qquad \xi_{(\psi)} = \frac{\partial}{\partial\psi}\,, \tag{2.9}$$

which can be used to define a family of locally nonrotating observers. The 5-velocity unit vector of these observers is given by

$$u^{\mu} = \alpha\left(\xi^{\mu}_{(t)} + \Omega_a\xi^{\mu}_{(\phi)} + \Omega_b\xi^{\mu}_{(\psi)}\right), \tag{2.10}$$

where α is determined by the condition $u^2 = -1$. The defining relations $u\cdot\xi_{(\phi)} = 0$ and $u\cdot\xi_{(\psi)} = 0$ allow us to determine the coordinate angular velocities Ω_a and Ω_b of the observers (see e.g. [29] for some details). Performing straightforward calculations, we find that

$$\Omega_a = \frac{\left(r^2+b^2\right)(2aM+bQ)-aQ^2}{\Delta\Sigma+2M\left(r^2+a^2\right)\left(r^2+b^2\right)-Q^2\left(r^2+a^2+b^2\right)}\,, \tag{2.11}$$

$$\Omega_b = \frac{\left(r^2+a^2\right)(2bM+aQ)-bQ^2}{\Delta\Sigma+2M\left(r^2+a^2\right)\left(r^2+b^2\right)-Q^2\left(r^2+a^2+b^2\right)}\,. \tag{2.12}$$

At large distances, as follows from these expressions, the bi-dragging property of the metric is governed by the remarkably simple formulas

$$\Omega_a = \frac{2aM+bQ}{r^4}+\mathcal{O}\left(\frac{1}{r^6}\right)\,, \qquad \Omega_b = \frac{2bM+aQ}{r^4}+\mathcal{O}\left(\frac{1}{r^6}\right). \tag{2.13}$$

We note that for vanishing rotation parameter $a=0$ (or $b=0$), the bi-dragging still occurs due to the electric charge of the black hole. The effect disappears at spatial infinity, while it increases towards the horizon and for $\Delta=0$, expressions (2.11) and (2.12) reduce to the angular velocities of the horizon [28]. We have

$$\Omega_{a(+)} = \frac{2\pi^2}{\mathcal{A}}\cdot\frac{a(r_+^2+b^2)+bQ}{r_+}\,, \qquad \Omega_{b(+)} = \frac{2\pi^2}{\mathcal{A}}\cdot\frac{b(r_+^2+a^2)+aQ}{r_+}\,, \tag{2.14}$$

where the horizon area $\mathcal{A}$ is given by

$$\mathcal{A} = \frac{2\pi^2\left[(r_+^2+a^2)(r_+^2+b^2)+abQ\right]}{r_+}\,. \tag{2.15}$$

With these quantities in mind, we can now introduce a co-rotating Killing vector defined as follows

$$\chi = \xi_{(t)}+\Omega_{a(+)}\,\xi_{(\phi)}+\Omega_{b(+)}\,\xi_{(\psi)}\,. \tag{2.16}$$

It is straightforward to show that the norm of this vector vanishes on the horizon, showing that it coincides with the null geodesic generators of the horizon. Using this vector, one can calculate the surface gravity κ of the horizon and hence its Hawking temperature T_H. We find that

$$T_H = \frac{\kappa}{2\pi} = \frac{\pi(r_+^2-r_-^2)}{\mathcal{A}}\,, \tag{2.17}$$

where we have used expressions (2.6) and (2.14). The co-rotating Killing vector can also be used to calculate the electrostatic potential of the horizon. Indeed, by means of potential one-form (2.4) and expressions (2.14), we find that the electrostatic potential of the horizon, relative to an infinitely distant observer, is given by

$$\Phi_H = -\,A\cdot\chi = \frac{\sqrt{3}\,\pi^2 Q r_+}{\mathcal{A}}\,. \tag{2.18}$$

Remarkably, the CCLP metric in (2.1) admits hidden symmetries which are generated by a second-rank Killing tensor [26, 30], in addition to its global symmetries given by Killing vectors (2.9). As a consequence, the geodesic and scalar field equations separate in this metric, ensuring their complete integrability. Below, we proceed with the separation of variables in the Klein-Gordon equation for a charged massless scalar field.

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3 Klein-Gordon equation

Let us consider a charged massless scalar field which obeys the Klein-Gordon equation $D^\mu D_\mu \Phi = 0$, where $D_\mu = \nabla_\mu - ieA_\mu$, and ∇_μ is the covariant derivative operator with respect to metric (2.1). Using expression (2.3) and the contravariant components of the metric given in (2.8), it is straightforward to show that this equation separates for the solution ansatz of the form

$$\Phi = e^{-i\omega t + im_\phi \phi + im_\psi \psi} S(\theta) R(r)\,, \tag{3.1}$$

where m_ϕ and m_ψ are "magnetic" quantum numbers associated with rotation in the ϕ and ψ directions. The angular function $S(\theta)$ obeys the equation

$$\frac{1}{\sin 2\theta}\frac{d}{d\theta}\left(\sin 2\theta \frac{dS}{d\theta}\right) + \left[\lambda - \omega^2\left(a^2 \sin^2\theta + b^2\cos^2\theta\right) - \frac{m_\phi^2}{\sin^2\theta} - \frac{m_\psi^2}{\cos^2\theta}\right] S = 0\,, \tag{3.2}$$

where we have used the freedom of shifting the separation constant, $\lambda \to \lambda + const$. As is known [31], this equation when accompanied by regular boundary conditions at singular points $\theta = 0$ and $\theta = \pi/2$ defines a Sturm-Liouville problem. The associated eigenvalues are $\lambda = \lambda_l(\omega)$, where l is an integer which can be thought of as an "orbital" quantum number. The solution is given by the five-dimensional spheroidal functions $S(\theta) = S_{\ell\, m_\phi m_\psi}(\theta|a\omega\,, b\omega)$, which form a complete set over the integer l. For nonvanishing rotation parameters, but for $a^2\omega^2 \ll 1$ and $b^2\omega^2 \ll 1$, one can show that

$$\lambda = l(l+2) + \mathcal{O}\left(a^2\omega^2, b^2\omega^2\right), \tag{3.3}$$

where l must obey the condition $l \geq m_\phi + m_\psi$ [31].

The radial equation for $R(r)$, by performing a few algebraic manipulations, can be cast in the form

$$\frac{\Delta}{r}\frac{d}{dr}\left(\frac{\Delta}{r}\frac{dR}{dr}\right) + U(r)\,R = 0\,, \tag{3.4}$$

where

$$U(r) = -\Delta\left[\lambda - 2\omega(am_\phi + bm_\psi) + \frac{(ab\,\omega - bm_\phi - am_\psi)^2}{r^2}\right] + \frac{\left[(r^2+a^2)(r^2+b^2) + ab\,Q\right]^2}{r^2} \times$$
$$\left\{\omega - \frac{m_\phi\left[a(r^2+b^2)+bQ\right]}{(r^2+a^2)(r^2+b^2)+abQ} - \frac{m_\psi\left[b(r^2+a^2)+aQ\right]}{(r^2+a^2)(r^2+b^2)+abQ} - \frac{\sqrt{3}}{2}\frac{eQr^2}{(r^2+a^2)(r^2+b^2)+abQ}\right\}^2. \tag{3.5}$$

For vanishing electric charge, $Q = 0$, these expressions go over into those obtained in [31]. They also agree with the vanishing cosmological constant limit of the expressions given in [26]. Next, it proves useful to transform the radial equation into a Schrödinger form. For this purpose, we introduce a new radial function $\mathcal{R}$ and a new radial coordinate r_*, which are defined by the relations

$$R = \left[\frac{r}{(r^2+a^2)(r^2+b^2)+abQ}\right]^{1/2}\mathcal{R}\,, \qquad \frac{dr_*}{dr} = \frac{(r^2+a^2)(r^2+b^2)+abQ}{\Delta}\,. \tag{3.6}$$

Using these definitions, we rewrite the radial equation (3.4) in the Schrödinger form

$$\frac{d^2\mathcal{R}}{dr_*^2} + V(r)\mathcal{R} = 0\,, \tag{3.7}$$

where the "effective" potential is given by

$$V(r) = -\frac{\Delta\left\{r^2\left[\lambda - 2\omega(am_\phi + bm_\psi)\right] + (ab\,\omega - bm_\phi - am_\psi)^2\right\}}{\left[(r^2+a^2)(r^2+b^2)+abQ\right]^2} - \frac{\Delta}{2rZ^{3/2}}\frac{d}{dr}\left(\frac{\Delta}{rZ^{3/2}}\frac{dZ}{dr}\right) + \left\{\omega - \frac{m_\phi\left[a(r^2+b^2)+bQ\right]}{(r^2+a^2)(r^2+b^2)+abQ} - \frac{m_\psi\left[b(r^2+a^2)+aQ\right]}{(r^2+a^2)(r^2+b^2)+abQ} - \frac{\sqrt{3}}{2}\frac{eQr^2}{(r^2+a^2)(r^2+b^2)+abQ}\right\}^2 \tag{3.8}$$

and we have also used the notation

$$Z = \left[(r^2+a^2)(r^2+b^2)+abQ\right]r^{-1}. \tag{3.9}$$

We are now interested in the behavior of the radial equation in the asymptotic regions, at spatial infinity $r_* \to \infty$ and at the horizon $r_* \to -\infty\,(r \to r_+)$, where the effective potential (3.8) takes the form

$$V(r) \to \begin{cases} \left(\omega - m_\phi\Omega_{a(+)} - m_\psi\Omega_{b(+)} - e\Phi_H\right)^2, & r_* \to -\infty \\ \omega^2, & r_* \to \infty \end{cases}. \tag{3.10}$$

Meanwhile, in the intermediate region it acts as a barrier, resulting in scattering processes of radial waves. In general, these asymptotic relations allow one to distinguish two classes of solutions to the radial wave equation: (i) the first class of solutions represents a wave originating at infinity (or being purely ingoing at the horizon), (ii) the second class of solutions corresponds to a wave originating in the past horizon (or being purely outgoing at infinity). The classical scattering process, which is the case under consideration, must be represented by the first class of solutions. That is, we have the following asymptotic behavior

$$\mathcal{R} \to \begin{cases} T_A\, e^{-i\left(\omega - m_\phi\Omega_{a(+)} - m_\psi\Omega_{b(+)} - e\Phi_H\right)r_*}, & r_* \to -\infty \\ e^{-i\omega r_*} + R_A\, e^{i\omega r_*}, & r_* \to \infty \end{cases} \tag{3.11}$$

where T_A and R_A are the transmission and reflection amplitudes respectively. The complex-conjugate of these asymptotic forms corresponds to the associated complex-conjugate solution of equation (3.7) as the effective potential $V(r)$ is a real quantity. Clearly, these two solutions are linearly independent and using the constancy of their Wronskian, we find that the transmission and reflection amplitudes obey the relation

$$|R_A|^2 = 1 - \frac{\omega - \omega_p}{\omega}\,|T_A|^2, \tag{3.12}$$

where have introduced the threshold frequency

$$\omega_p = m_\phi\Omega_{a(+)} + m_\psi\Omega_{b(+)} + e\Phi_H. \tag{3.13}$$

It follows that for the frequency range given by the inequality

$$0 < \omega < \omega_p \,, \tag{3.14}$$

the reflected wave has greater amplitude than the incident one, $|R_A|^2 > 1$, i.e. the superradiant effect appears. We note that the presence of the electric charge changes the threshold frequency of superradiance. This occurs not only due to the nonvanishing electrostatic potential of the horizon but also because of the gravimagnetic bi-dragging contribution to its angular velocities.

4 Solutions

The analysis of singularity structure of the radial equation (3.4) reveals that solutions to this equation possess an essential singularity. This means that one can not use the familiar techniques, employed in the theory of ordinary linear differential equations, to find the general solutions to this equation (see e.g. [32]). On the other hand, one can certainly find such solutions to some approximated versions of this equation, which are applicable in various regions of the spacetime. In what follows, we are interested in solutions at low frequencies i.e., when the Compton wavelength of the scalar particle is much larger than the horizon radius of the black hole. Following the work of Starobinsky [11], we divide the spacetime into the near-horizon and far regions and approximate equation (3.4) for each of these regions. Solving then the resulting equations with appropriate boundary conditions, we assume that the orbital quantum number l is nearly integer, thereby avoiding the appearance of solutions with logarithmic terms. This allows us to perform the matching of the solutions under consideration in the overlap between the near and far regions and thus obtaining the complete solution at low frequencies. Below, we discuss these equations and solutions to them as well as the matching procedure in the overlap region.

4.1 Near-region

In the region near the horizon, $r - r_+ \ll 1/\omega$, and for low-frequency perturbations $r_+ \ll 1/\omega$, equation (3.4) takes the form

$$\frac{\Delta}{r}\frac{d}{dr}\left(\frac{\Delta}{r}\frac{dR}{dr}\right) + \left(\frac{\omega - \omega_p}{2\pi^2}\,\mathcal{A}\right)^2 R - l(l+2)\Delta R = 0\,, \tag{4.1}$$

where we have used relations (2.15) and (3.3), assuming slow rotation as well. For future purposes, we will henceforth assume that l is nearly integer, thus keeping in mind small corrections in (3.3) and (3.5). Next, using a new dimensionless variable

$$z = \frac{r^2 - r_+^2}{r^2 - r_-^2}\,, \tag{4.2}$$

one can show that equation (4.1) reduces to the hypergeometric type equation

$$z(1-z)\frac{d^2R}{dz^2} + (1-z)\frac{dR}{dz} + \left[\frac{1-z}{z}\,\Omega^2 - \frac{\ell(\ell+2)}{4(1-z)}\right] R = 0\,, \tag{4.3}$$

where

$$\Omega = \frac{\omega - \omega_p}{4\pi T_H}\,. \tag{4.4}$$

This equation can be solved in a standard way by the ansatz

$$R(z) = z^{i\Omega}\,(1-z)^{1+l/2}\,F(z)\,, \tag{4.5}$$

where $F(z) = F(\alpha\,,\beta\,,\gamma, z)$ is the hypergeometric function, obeying the equation

$$z(1-z)\frac{d^2F}{dz^2} + \left[\gamma - (\alpha+\beta+1)\,z\right]\frac{dF}{dz} - \alpha\beta F = 0\,, \tag{4.6}$$

and the parameters are given by

$$\alpha = 1 + l/2 + 2i\Omega\,, \qquad \beta = 1 + l/2\,, \qquad \gamma = 1 + 2i\Omega\,. \tag{4.7}$$

Thus, the general solution to equation (4.3) can be written in terms of two linearly independent solutions of equation (4.6). We need the physical solution that reduces to the ingoing wave at the horizon, $z \to 0$. It is given by

$$R(z) = A^{\rm in}_{(+)}\,z^{-i\Omega}\,(1-z)^{1+l/2}\,F\left(1+l/2\,,1+l/2-2i\Omega\,,1-2i\Omega\,,z\right)\,, \tag{4.8}$$

where $A^{\rm in}_{(+)}$ is a constant. Furthermore, in an overlapping region the large r behavior of this solution should be compared with the small r behavior of the far-region solution. Therefore, we also need the large r ($z \to 1$) limit of solution (4.8) which can be easily found by using the pertinent modular properties of the hypergeometric functions [33]. We find that the large r behavior of the near-horizon region solution is given by

$$R \sim A^{\rm in}_{(+)}\,\Gamma(1-2i\Omega)\left[\frac{\Gamma(-l-1)\,(r_+^2-r_-^2)^{1+l/2}}{\Gamma(-l/2)\,\Gamma(-l/2-2i\Omega)}\,r^{-2-l} + \frac{\Gamma(l+1)\,(r_+^2-r_-^2)^{-l/2}}{\Gamma(1+l/2)\,\Gamma(1+l/2-2i\Omega)}\,r^{l}\right]. \tag{4.9}$$

It is important to note that in this expansion the first term inside the square bracket requires a special care for l approaching the integer values as the quotient of two gamma functions $\Gamma(-l-1)/\Gamma(-l/2)$ becomes divergent for some values of l. We will return to this issue in more detail below.

4.2 Far-region

In the far-region, $r - r_+ \gg r_+$, equation (3.4) can be approximated by

$$\frac{d^2R}{dr^2} + \frac{3}{r}\frac{dR}{dr} + \left[\omega^2 - \frac{\ell(\ell+2)}{r^2}\right]R = 0\,. \tag{4.10}$$

Here l is again supposed to be nearly, but not exactly, integer by taking into account small corrections in the region under consideration, including the Newtonian term $\sim \omega^2 r_+^2/r^2$ in five dimensions.

Using the ansatz $R = u/r$ and rescaling the radial variable as $x = \omega r$, one can show that equation (4.10) reduces to the standard Bessel equation given by

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - \left[x^2 - (l+1)^2\right]u = 0\,. \tag{4.11}$$

As is known [33], the general solution of this equation is a linear combination of the Bessel and Neumann functions. We have

$$R(r) = \frac{1}{r}\left[A_\infty\,J_{l+1}(\omega r) + B_\infty\,N_{l+1}(\omega r)\right], \tag{4.12}$$

where A_∞ and B_∞ are constants. Though this solution refers only to large r region, but for small x ($\omega r \ll 1$) it might also have a limiting behavior, which indicates on an overlapping regime of validity with the large r form of the near-horizon solution (4.9). For small ωr, using the asymptotic forms of the Bessel and Neumann functions, we find that

$$R(r) \sim A_\infty \left(\frac{\omega}{2}\right)^{l+1} \frac{r^l}{\Gamma(l+2)} - B_\infty \left(\frac{2}{\omega}\right)^{l+1} \frac{\Gamma(l+1)}{\pi}\, r^{-2-l}\,. \tag{4.13}$$

For some further purposes, it may also be useful to know the large ωr behavior of solution (4.12), which is given by

$$R(r) \sim \frac{1}{\sqrt{2\pi\omega r^3}} \left[(A_\infty + iB_\infty)\, e^{\frac{i\pi}{2}(l+\frac{3}{2})} e^{-i\omega r} + (A_\infty - iB_\infty)\, e^{-\frac{i\pi}{2}(l+\frac{3}{2})} e^{i\omega r}\right], \tag{4.14}$$

where, as expected, the first term refers to an ingoing wave and the second term corresponds to an outgoing wave.

4.3 Matching procedure

With the above discussion of solutions, referring to the near-horizon and far regions of the spacetime, it becomes clear that the construction of the complete low-frequency solution for the radial waves requires a matching procedure in an intermediate region. Before doing this several comments are in order. Since the gamma function develops the pole structure when its argument is a negative integer, it is easy to see that the quotient of gamma functions $\Gamma(-l-1)/\Gamma(-l/2)$ appearing in expression (4.9) diverges for odd integer values of l. Consequently, solutions with logarithmic terms will inevitably appear. This makes the matching procedure impossible for odd l, as noted in [27]. However, assuming that l is not exactly, but nearly integer one can avoid the appearance of solutions with logarithmic terms and proceed with the matching procedure. This is the reason why we introduce the “nearly integer” l in the above description of the solutions (see also [34]).

With this in mind, we compare equations (4.9) and (4.13) and see that there exists an overlapping regime of validity ($r_+ \ll r - r_+ \ll 1/\omega$) for the near-horizon and far region solutions. Performing the matching in this regime, we find that the defining amplitude ratios are given by

$$\frac{A^{\rm in}_{(+)}}{A_\infty} = \left(\frac{\omega}{2}\right)^{l+1} \frac{(r_+^2 - r_-^2)^{l/2}\,\Gamma(1+l/2)}{\Gamma(l+1)\Gamma(l+2)}\, \frac{\Gamma(1+l/2-2i\Omega)}{\Gamma(1-2i\Omega)}\,, \tag{4.15}$$

$$\frac{B_\infty}{A_\infty} = -\pi \left(\frac{\omega}{2}\right)^{2(l+1)} \frac{(r_+^2 - r_-^2)^{l+1}\,\Gamma(1+l/2)}{\Gamma^2(l+1)\Gamma(l+2)}\, \frac{\Gamma(-l-1)}{\Gamma(-l/2)}\, \frac{\Gamma(1+l/2-2i\Omega)}{\Gamma(-l/2-2i\Omega)}\,. \tag{4.16}$$

We are now in position to proceed with the superradiant instability of the rotating black hole by placing a reflecting mirror around it.

5 Reflecting mirror and negative damping

As we have described in the introduction, one of the most striking application of the superradiant effect in four dimensions amounts to exploring the black hole-mirror system, which under certain condition acts as a black hole bomb [10]. In this section, we wish to explore this phenomenon in five dimensions, using the model which consists of a rotating black hole of minimal ungauged supergravity [28] and a reflecting mirror located at a large distance L from the black hole $(L \gg r_+)$. We assume that the mirror perfectly reflects low-frequency scalar waves, so that on the surface of the mirror one must impose the vanishing field condition. This, by equation (4.12), yields

$$A_\infty \, J_{l+1}(\omega L) + B_\infty \, N_{l+1}(\omega L) = 0 \,. \tag{5.1}$$

This condition, when combined with that requiring a purely ingoing wave at the horizon, defines a characteristic-value problem for the confined spectrum of the low-frequency solution, discussed above. Such a spectrum would be quasinormal with complex frequencies whose imaginary part describes the damping of modes, as can be seen from equation (3.1). When the imaginary part is positive, a characteristic mode undergoes exponential growth (*the negative damping*). In this case, the system will develop instability, creating a black hole bomb.

Comparing now equations (4.16) and (5.1), we obtain the defining transcendental equation for the frequency spectrum

$$\frac{J_{l+1}(\omega L)}{N_{l+1}(\omega L)} = \pi \left(\frac{\omega}{2}\right)^{2(l+1)} \frac{(r_+^2 - r_-^2)^{l+1}\,\Gamma(1+l/2)}{\Gamma^2(l+1)\Gamma(l+2)} \, \frac{\Gamma(-l-1)}{\Gamma(-l/2)} \, \frac{\Gamma(1+l/2-2i\Omega)}{\Gamma(-l/2-2i\Omega)} \,, \tag{5.2}$$

which can be solved by iteration in the low-frequency approximation. Let us assume that the solution to this equation can be written in the form

$$\omega = \omega_n + i\delta \,, \tag{5.3}$$

where n is a non-negative integer, ω_n describes the discrete frequency spectrum of free modes and δ is supposed to be a small damping parameter, representing a "response" to the ingoing wave condition at the horizon. Using this in equation (5.2) it is easy to see that, in lowest approximation, ω_n is simply given by the real roots of the Bessel function. Thus, we have

$$\omega_n = \frac{j_{l+1\,,\,n}}{L} \,, \tag{5.4}$$

where the quantity $j_{l+1\,,\,n}$ represents the n-th root (greater than zero) of the equation $J_{l+1}(\omega_n L) = 0$. A detailed list of these roots can be found in [33]. They can also be easily tabulated using Mathematica. On the other hand, for large overtones of the fundamental frequency $(n \gg 1)$ one can appeal to the asymptotic form of the Bessel function, which gives the simple formula

$$j_{l+1\,,\,n} \simeq \pi \,(n + l/2) \;. \tag{5.5}$$

It should be noted that formula (5.4) generalizes to five dimensions the familiar flat spacetime result for the frequency spectrum in an infinitely deep spherical potential well [35].

Next, substituting equations (5.3) and (5.4) in equation (5.2) and performing a few algebraic manipulations, to first order in δ, we find that the damping parameter is given by

$$\delta = -\frac{i\pi}{L}\,\frac{N_{l+1}(j_{l+1\,,\,n})}{J'_{l+1}(j_{l+1\,,\,n})} \left(\frac{j_{l+1\,,\,n}}{2L}\right)^{2(l+1)} \frac{(r_+^2 - r_-^2)^{l+1}\,\Gamma(1+l/2)}{\Gamma^2(l+1)\,\Gamma(l+2)} \times$$
$$\frac{\Gamma(-l-1)}{\Gamma(-l/2)}\,\frac{\Gamma(1+l/2-2i\Omega)}{\Gamma(-l/2-2i\Omega)} \,. \tag{5.6}$$

Here the prime denotes the derivative of the Bessel function with respect to its argument and the quantity Ω, as follows from equation (4.4), is given by

$$\Omega = \frac{r_+^3}{2} \frac{\omega_n - \omega_p}{r_+^2 - r_-^2} \,. \tag{5.7}$$

Comparing this expression with that given in (5.4), we see that the superradiant effect crucially depends on the distance L at which the mirror is placed just as in four dimensions [24]. That is, for a critical distance governing the fundamental frequency, the effect ceases to exist. To proceed further, it is useful to simplify separately the product of the quotients of gamma functions in the second line of equation (5.6). Using the well known relation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, it is straightforward to show that

$$\frac{\Gamma(-l-1)}{\Gamma(-l/2)} = -\frac{1}{2\cos(\pi l/2)} \frac{\Gamma(1+l/2)}{\Gamma(l+2)} \,, \tag{5.8}$$

$$\begin{aligned} \frac{\Gamma(1+l/2-2i\Omega)}{\Gamma(-l/2-2i\Omega)} &= -\frac{1}{\pi} \left|\Gamma(1+l/2-2i\Omega)\right|^2 \left[\sin(\pi l/2)\cosh(2\pi\Omega)\right. \\ &\quad \left. + \, i\cos(\pi l/2)\sinh(2\pi\Omega)\right] . \end{aligned} \tag{5.9}$$

Substituting now these relations in equation (5.6), we have

$$\begin{aligned} \delta = \frac{i}{2L} \left| \frac{N_{l+1}(j_{l+1\,,\,n})}{J'_{l+1}(j_{l+1\,,\,n})} \right| \left(\frac{j_{l+1\,,\,n}}{2L} \right)^{2(l+1)} \frac{(r_+^2 - r_-^2)^{l+1} \Gamma^2(1+l/2)}{\Gamma^2(l+1)\,\Gamma^2(l+2)} \times \\ \frac{\left|\Gamma(1+l/2-2i\Omega)\right|^2}{\cos(\pi l/2)} \left[\sin(\pi l/2)\cosh(2\pi\Omega) + i\cos(\pi l/2)\sinh(2\pi\Omega)\right] , \end{aligned} \tag{5.10}$$

where we have changed the overall sign, taking the absolute value of the quotient $\frac{N_{l+1}(j_{l+1\,,\,n})}{J'_{l+1}(j_{l+1\,,\,n})}$, since it is always negative in the physically acceptable frequency range. Recalling that here l is nearly integer, we can further simplify this equation by specifying l. Let us now assume that l approaches either even or odd integers. That is, we consider the following cases;

(i) $l/2 = p + \epsilon$, where p is a non-negative integer and $\epsilon \to 0$. Substituting this in expression (5.10), we find that its imaginary part vanishes in the limit $\epsilon \to 0$, whereas the real part is given by

$$\delta = -\pi\Omega \left| \frac{N_{2p+1}(j_{2p+1\,,\,n})}{J'_{2p+1}(j_{2p+1\,,\,n})} \right| \left(\frac{j_{2p+1\,,\,n}}{2L} \right)^{2(2p+1)} \frac{(r_+^2 - r_-^2)^{2p+1}}{L} \left(\frac{p!}{(2p)!(2p+1)!} \right)^2 \prod_{k=1}^{p} \left(k^2 + 4\Omega^2\right) . \tag{5.11}$$

In obtaining this expression we have used the identity

$$\left|\Gamma(1+l/2-2i\Omega)\right|^2 = \frac{2\pi\Omega}{\sinh(2\pi\Omega)} \prod_{k=1}^{p} \left(k^2 + 4\Omega^2\right) , \tag{5.12}$$

which can be easily obtained from the pertinent properties of gamma functions [33]. It should be noted that indeed in the case under consideration, there are no divergencies in expression (5.10) when $\epsilon \to 0$, so that throughout the calculations one can simply set ϵ equal to zero. Turning back to equation (5.11), we see that its sign is entirely determined by the sign of the quantity Ω, becoming positive in the superradiant regime, $\Omega < 0$. Thus, for all modes of even l we have the negative damping effect, resulting in exponential growth of their amplitudes.

(ii) $l/2 = (p + 1/2) + \epsilon$, again p is a non-negative integer and $\epsilon \to 0$. Inserting this in expression (5.10), we need to consider the limit as $\epsilon \to 0$. After performing a few straightforward calculations, we obtain that

$$\delta = -\left|\frac{N_{2p+2}(j_{2p+2\,,\,n})}{J'_{2p+2}(j_{2p+2\,,\,n})}\right| \left(\frac{j_{2p+2\,,\,n}}{2L}\right)^{2(2p+2)} \frac{(r_+^2 - r_-^2)^{2p+2}}{2L} \frac{\Gamma^2(p+3/2)}{\Gamma^2(2p+2)\Gamma^2(2p+3)} \times$$
$$|\Gamma(p+3/2-2i\Omega)|^2 \left(\sinh(2\pi\Omega) + \frac{i}{\epsilon}\,\frac{\cosh(2\pi\Omega)}{\pi}\right). \tag{5.13}$$

Using the properties of gamma functions [33], resulting in the relations

$$\Gamma\left(p+\frac{1}{2}\right) = \frac{\pi^{1/2}\,(2p)!}{2^{2p}\,p!}\,, \tag{5.14}$$

$$|\Gamma(p+3/2-2i\Omega)|^2 = \frac{\pi}{\cosh(2\pi\Omega)} \prod_{k=1}^{p+1}\left[(k-1/2)^2 + 4\Omega^2\right], \tag{5.15}$$

one can further simplify the combination of gamma functions appearing in equation (5.13). Finally, we have

$$\delta = -\pi\left|\frac{N_{2p+2}(j_{2p+2\,,\,n})}{J'_{2p+2}(j_{2p+2\,,\,n})}\right| \left(\frac{j_{2p+2\,,\,n}}{4L}\right)^{2(2p+2)} \frac{(r_+^2 - r_-^2)^{2p+2}}{2L} \times$$
$$\frac{(\pi\tanh(2\pi\Omega) + i/\epsilon)}{[(p+1)!(2p+1)!]^2} \prod_{k=1}^{p+1}\left[(k-1/2)^2 + 4\Omega^2\right]. \tag{5.16}$$

It is easy see that this expression possesses two important features: first, its real part that describes the damping of the modes changes the sign in the superradiant regime, $\Omega < 0$. This means that all modes of odd l may become supperradiant as well, resulting in the instability of the system. Meanwhile, the sign changing does not occur for the imaginary part, which is not sensitive to superradiance at all. Second, the imaginary part involves $1/\epsilon$ type divergence as $\epsilon \to 0$. However, this divergence can somewhat be smoothed out by using the fact that the quantity r_+ is indeed small, in accordance with the regime of validity of the low-frequency solution constructed above. Thus, for a given radius L of the mirror and for the lowest mode ($p = 0$), the ratio $(r_+^2 - r_-^2)^2/\epsilon$ appearing in the imaginary part can be fixed as finite, to high accuracy. The accuracy considerably increases for higher modes, as can be seen from (5.16). This would result in a small frequency-shift in the spectrum. These arguments is further supported by a numerical analysis of expression (5.16).

In table 1 we present the numerical results for a charged nonrotating black hole. For the extreme charge of the black hole, we have $Q_e = r_+^2$, as follows from expressions (2.6)

q	$\delta_{\ell/2=p+\epsilon}\,,\quad \epsilon\to 0$	$\delta_{\ell/2=(p+1/2)+\epsilon}\,,\quad \epsilon\to 10^{-7}$
0.1	-5.653×10^{-5}	$-2.992\times10^{-8}-0.462\times10^{-7}\,\mathrm{i}/\epsilon$
0.3	-4.035×10^{-5}	$-2.248\times10^{-8}-0.422\times10^{-7}\,\mathrm{i}/\epsilon$
0.5	-2.274×10^{-5}	$-1.408\times10^{-8}-0.346\times10^{-7}\,\mathrm{i}/\epsilon$
0.7	-3.911×10^{-6}	$-5.802\times10^{-9}-0.234\times10^{-7}\,\mathrm{i}/\epsilon$
0.8	$+5.956\times10^{-6}$	$-2.213\times10^{-9}-0.165\times10^{-7}\,\mathrm{i}/\epsilon$
0.9	$+16.221\times10^{-6}$	$+4.965\times10^{-10}-0.087\times10^{-7}\,\mathrm{i}/\epsilon$

Table 1. The damping parameter of quasinormal modes ($p=0,\, n=1$); the scalar field charge $e=10,$ the black hole parameters $a=b=0,\; r_+=0.01$ and $q=Q/Q_e\,.$

α	$\delta_{\ell/2=p+\epsilon}\,,\quad \epsilon\to 0$	$\delta_{\ell/2=(p+1/2)+\epsilon}\,,\quad \epsilon\to 10^{-7}$
0.1	4.673×10^{-13}	$5.513\times10^{-9}-0.115\times10^{-7}\,\mathrm{i}/\epsilon$
0.2	1.788×10^{-12}	$1.708\times10^{-8}-0.124\times10^{-7}\,\mathrm{i}/\epsilon$
0.3	3.215×10^{-12}	$2.935\times10^{-8}-0.143\times10^{-7}\,\mathrm{i}/\epsilon$
0.33	3.675×10^{-12}	$3.323\times10^{-8}-0.150\times10^{-7}\,\mathrm{i}/\epsilon$

Table 2. The damping parameter of quasinormal modes with $m_\phi=1$ ($p=1,\, n=1$ in the even l case, while $p=0,\, n=1$ in the odd l case); the black hole parameters $r_+=0.01,\;\; \alpha=a/r_+\,,\;\; b=0$ and $Q=0$.

and (2.7), and we take $L=1$, for certainty. The calculations are performed for the parameters $r_+=0.01,\;\; e=10$ and for the lowest modes as l approaches even or odd integers. We see that the superradiant instability appears in both cases, when the charge of the black hole is close to the extreme value. Meanwhile, for $\epsilon\to 10^{-7}$, the imaginary part of the damping parameter (in the odd l case) represents a small frequency-shift in the spectrum. Table 2 gives a summary of the numerical analysis of the damping parameter for a singly rotating black hole with zero electric charge, $Q=0$. It follows that the superradiant instability occurs to all l modes of scalar perturbations under consideration. Again, we a have a small frequency-shift for the $l=1$ mode, by choosing $\epsilon\to 10^{-7}$.

Thus, we conclude that in the black hole-mirror model under consideration, all l modes of scalar perturbations become unstable in the regime of superradiance, exponentially growing their amplitudes with characteristic time scale $\tau=1/\delta$. In addition, the modes of odd l undergo small frequency-shifts in the spectrum.

6 Conclusion

The superradiant instabilities of black hole-mirror systems as well as small AdS black holes in four-dimensional spacetimes have been extensively studied in [24, 25] by employing both analytical and numerical approaches. The analytical approach is based on a matching procedure, first introduced by Starobinsky [11], that allows one to find the complete low-frequency solution to the Klein-Gordon equation by matching the near-horizon and far regions solutions in their overlap region. In our earlier work [26], using a similar analytical approach we gave a quantitative description of the superradiant instability of small rotating charged AdS

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black holes in five dimensions. In a recent development [27], this investigation was continued for small Reissner-Nordström-AdS black holes in all spacetime dimensions. Here it was also pointed out that in odd spacetime dimensions, the matching procedure used in [26] fails for some values of the orbital quantum number l, thus making the use of numerical methods inevitable.

The purpose of this paper was to embark on a further investigation of the superradiant instability in five dimensions, elaborating on the black hole bomb model which consists of a rotating black hole of five-dimenensional minimal ungauged supergravity [28] and a reflecting mirror around it. In spite of some subtleties with the matching procedure in five dimensions, we have shown that one can still successfully use the analytical approach to give the quantitative description of the black hole bomb model under consideration.

Our results can be summarized as follows: after demonstrating the full separability of the Klein-Gordon equation, we have discussed the behavior of the radial wave equation in the asymptotic regions and derived the threshold inequality for superradiance. Next, focusing on low-frequency perturbations and slow rotation, we have approximated the radial wave equation in the near-horizon and far regions of the spacetime and solved the resulting equations with appropriate boundary conditions in each of these regions separately. To avoid the appearance of solutions with logarithmic terms, which do not comply with the matching procedure, we have assumed that the orbital quantum number l is not exactly, but nearly integer. With this in mind, we have performed the matching of the near-horizon and far regions solutions in an intermediate region, thereby constructing the complete low-frequency solution to the Klein-Gordon equation.

In the black hole-mirror system, we have defined a characteristic-value problem for the confined (quasinormal) spectrum of the low-frequency solution and calculated the complex frequencies of the spectrum. We have found the general expression for the imaginary part (for the small damping parameter) of the quasinormal spectrum, which appeared to be a complex quantity. Next, taking the limit as l approaches an even integer, we have shown that the imaginary part of the damping parameter vanishes identically, whereas its real part becomes positive in the superradiant regime. Thus, all modes of even l undergo negative damping, resulting in exponential growth of their amplitudes. Meanwhile, in the limit as l approaches an odd integer, the damping parameter remains complex whose real part is positive in the superradiant regime, thereby showing that all modes of odd l become unstable as well. As for the imaginary part, its sign appears to be not sensitive to superradiance at all. We have argued that to high accuracy, the imaginary part of the damping parameter can be considered as representing a small frequency-shift in the spectrum, as discussed at the end of section 5.

Finally, we have concluded that that in the five-dimensional black hole-mirror system, all l modes of scalar perturbations undergo negative damping in the regime of superradiance, exponentially growing their amplitudes and thus creating the black hole bomb effect in five dimensions.

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